Generalized ideals in semigroups

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Introduction

A semigroup is a non-empty set in which an associative binary multiplication is defined. A semigroup S is commutative, if $ab=ba$ for all $a, b \in S$. Let A, B be arbitrary non-empty subsets of a semigroup S, then the product AB means the set of all elements ab ($a \in A, b \in B$). A subsemigroup of S is a non-empty subset T of S, which forms a semigroup under the same operation as S. A subset T is a subsemigroup if and only if $TT \subseteq T$ holds. A left (right) ideal L (R) of S is a non-empty subset of S such that $SL \subseteq L$ ($RS \subseteq R$) holds. A two-sided ideal or ideal of S is a subset which is both a left and a right ideal of S. The smallest left (right, two-sided) ideal of S containing the subset A is called the left (right, two-sided) ideal of S generated by A. If an ideal is generated by a single element, it is termed principal.

In this paper we introduce the following generalizations of the concepts of ideals in semigroups: the notion of (m, n)-ideal, which is a generalization of one-sided (left or right) ideals, and as a special case it contains the notion of biideal, due to R. A. GOOD and D. R. HUGHES [1]; the concept of (m, n)-quasiideal, which is a generalization of the concept of quasiideal, due to O. STEINFELD [5]; the concept of k-ideal, which is a generalization of the concept of two-sided ideal.

We note that these notions can be introduced in an arbitrary algebraic system, in which at least one associative operation is defined.

§ 1. (m, n)-ideals

Definition 1.1. We call the subsemigroup A of an arbitrary semigroup S an *(m, n)-ideal,* if A satisfies the relation

(1) $$A^m S A^n \subseteq A,$$

where m, n are non-negative integers. (A^0 let be defined as an operator element, so that $A^0 S = SA^0 = S$.)

R. A. GOOD and D. R. HUGHES [1] have introduced the notion of (1, 1)-ideal under the name „biideal".

It is easy to prove the following properties of (m, n)-ideals:

a) The intersection of two (m, n)-ideals of a semigroup S is the empty set, or else an (m, n)-ideal of S.

b) A group has no proper (m, n)-ideal.

c) Let k be a positive integer; the k-th power of an (m, n)-ideal is also an (m, n)-ideal.

d) Let A be a subset of a semigroup S. The smallest (m, n)-ideal of S containing A is called the (m, n)-ideal of S generated by A and denoted by $\{A\}_{(m,n)}$. It is clear that

$$\{A\}_{(m,n)} = A \cup A^2 \cup \cdots \cup A^{m+n} \cup A^m S A^n.$$

e) If A is a subsemigroup of S, then

$$\{A\}_{(m,n)} = A \cup A^m S A^n.$$

f) The principal (m, n)-ideal of S generated by element a of S is

$$a \cup a^2 \cup \cdots \cup a^{m+n} \cup a^m S a^n.$$

g) The principal (m, n)-ideal of S generated by an idempotent element e is eSe.

Definition 1.2. A subsemigroup S_n of a semigroup S will be called *attainable*, if there exist subsemigroups $S_1, S_2, \ldots, S_{n-1}$ of S such that

$$S_n \subseteq S_{n-1} \subseteq \cdots \subseteq S_1 \subseteq S_0 = S \tag{2}$$

holds, where S_i is a one-sided (left or right) ideal of S_{i-1} $(i = 1, 2, \ldots, n)$.

With every such chain (2) of subsemigroups we can associate a product π of the letters l and r in which the i-th factor is l or r according to whether S_i $(i = 1, \ldots, n)$ is contained in S_{i-1} as a left or right ideal, respectively. (If S_i is a two-sided ideal in S_{i-1}, then either of l and of r can be chosen.) A subsemigroup A of S is called a *π-ideal*, if it is attainable by a subsemigroup chain with which the product π is associated.

In the product π let m and n be the numbers of the factors l and r, respectively.

We are going to prove the following

Theorem 1.3. *The following three statements concerning to a subset A of an arbitrary semigroup S are equivalent:*

(i) *A is an lr-ideal of S,*

(ii) *A is an rl-ideal of S,*

(iii) *A is an $(1, 1)$-ideal of S.*

Proof. Let A be an lr-ideal of a semigroup S. Then for some subsemigroup L of S we have $A \subseteq L \subseteq S$, $AL \subseteq A$ and $SL \subseteq L$. Hence it follows that

$$ASA \subseteq ASL \subseteq AL \subseteq A,$$

that is, A is an $(1, 1)$-ideal of S.

Conversely, let A be $(1, 1)$-ideal of a semigroup S, i. e. $ASA \subseteq A$. Then by

$$A(A \cup SA) = AA \cup ASA \subseteq A \cup A = A$$

it follows that A is a right ideal of the left ideal of S generated by A, i. e. A is an lr-ideal of S.

The proof of the dual statement is similar.

Corollary 1.4. *A subset A of a semigroup S is a π-ideal of S if and only if A is an $r^m l^n$-ideal of S.*

Now we can prove the

Theorem 1.5. *A subset A of a semigroup S is a π-ideal of S if and only if A is an (m, n)-ideal of S.*

Proof. By the preceding corollary it suffices to show the theorem for $r^m l^n$-ideals instead of π-ideals. Let A be an $r^m l^n$-ideal of a semigroup S. Then — A being an attainable subsemigroup — there exist subsemigroups $L_1, L_2, \dots, L_{n-1}$, and $R_1, R_2, \dots, R_m$ of S such that the following relations hold:

$$(3) \quad \begin{cases} A = L_n \subseteq L_{n-1} \subseteq \cdots \subseteq L_1 \subseteq R_m \subseteq \cdots \subseteq R_1 \subseteq R_0 = S, \\ R_i R_{i-1} \subseteq R_i \ (i = 1, \dots, m), \quad R_m L_1 \subseteq L_1, \quad L_{j-1} L_j \subseteq L_j \ (j = 2, \dots, n). \end{cases}$$

Hence it follows that

$$A^m S A^n = L_n^m S L_n^n \subseteq L_n^{m-1} (R_1 S) L_n^n \subseteq L_n^{m-1} R_1 L_n^n \subseteq \cdots \subseteq (L_n R_{m-1}) L_n^n \subseteq$$
$$\subseteq (R_m R_{m-1}) L_n^n \subseteq (R_m L_1) L_n^{n-1} \subseteq L_1 (L_2 L_n^{n-2}) \subseteq \cdots \subseteq L_n = A,$$

therefore A is indeed an (m, n)-ideal of S.

Conversely, let us suppose that A is an (m, n)-ideal of the semigroup S. By the property *e*) the (m, n)-ideal of S generated by A is $A \cup A^m S A^n$. It is easy to see that $\{A\}_{(m, k)}$ is a left ideal of $\{A\}_{(m, k-1)}$ $(k = 1, \dots, n)$, and $\{A\}_{(i, n)}$ is a right ideal in $\{A\}_{(i-1, n)}$ $(i = 1, \dots, m)$. Hence the subsemigroups $L_n = A$, $L_{n-1} = \{A\}_{(m, n-1)}, \dots, L_1 = \{A\}_{(m, 1)}$, $R_m = \{A\}_{(m, 0)}$, $R_{m-1} = \{A\}_{(m-1, 0)}, \dots$ $\dots, R_1 = \{A\}_{(1, 0)}$ satisfy the conditions (3). Thus A is an $r^m l^n$-ideal of S. This completes the proof of Theorem 1.5.

Definition 1.6. A two-sided ideal of a two-sided ideal of a semigroup S we shall call an i^2-ideal. By an *i^k-ideal* we mean a two-sided ideal of an arbitrary i^{k-1}-ideal of S, where k is a positive integer $(k \geqq 2)$.

Definition 1.7. By a *k-ideal* of a semigroup S we mean a subset A which is an (m, n)-ideal of S, for every m, n such that $m+n=k$.

It is clear that the subset A of a commutative semigroup S is a k-ideal if and only if $A^k S \subseteq A$. We remark that the concept of k-ideal is a generalization of the concept of two-sided ideal.

Corollary 1.8. *The subset A of a commutative semigroup S is an i^k-ideal if and only if it is a k-ideal.*

Proof. This follows at once from Theorem 1.5.

Remark 1.9. More generally the Corollary 1.8 holds for two-sided semigroups too. By a two-sided (or duo) semigroup we mean a semigroup every one-sided ideal in which is a two-sided ideal (see: [4]).

§ 2. (m, n)-quasiideals

Definition 2.1. A subsemigroup A of a semigroup S we shall call an *(m, n)-quasiideal,* if

$$A^m S \cap S A^n \subseteq A \tag{4}$$

holds, where m, n are non-negative integers (A^0 is an operator element not contained in S, and $A^0 S = S A^0 = S$).

It is easy to prove the following properties of (m, n)-quasiideals:

a) The intersection of a set of (m, n)-quasiideals of a semigroup S, if it is not empty, is an (m, n)-quasiideal of S.

b) A group has no proper (m, n)-quasiideal.

c) Let A be a subset of a semigroup S. The (m, n)-quasiideal of S generated by A, i. e. the smallest (m, n)-quasiideal of S containing A is

$$A \cup A^2 \cup \cdots \cup A^k \cup (A^m S \cap S A^n),$$

where $k = \min(m, n)$.

d) If A is a subsemigroup of S, then the (m, n)-quasiideal of S generated by A is

$$A \cup (A^m S \cap S A^n).$$

e) The principal (m, n)-quasiideal generated by a is

$$a \cup a^2 \cup \cdots \cup a^k \cup (a^m S \cap S a^n),$$

where $k = \min(m, n)$.

f) The principal (m, n)-quasiideal generated by an idempotent element e is $eS \cap Se$.

The concept of (1, 1)-quasiideal was introduced by O. STEINFELD [5] under the name „quasiideal", and he showed that a subset of a semigroup S is a quasiideal, if and only if it is an intersection of a left and a right ideal of S. This result is generalized by the

Theorem 2.2. *A subset of an arbitrary semigroup S is an (m, n)-quasiideal of S if and only if it is the intersection of an $(m, 0)$-ideal and a $(0, n)$-ideal of S.*

Proof. Let S be an arbitrary semigroup, let A and B be an $(m, 0)$-ideal and a $(0, n)$-ideal of S, respectively. Then $A^m S \subseteq A$, and $SB^n \subseteq B$, which implies

$$(A \cap B)^m S \cap S(A \cap B)^n \subseteq A \cap B,$$

and since the common part of subsemigroups is likewise a subsemigroup, we obtain that $A \cap B$ is an (m, n)-quasiideal of S.

Conversely, let A be an (m, n)-quasiideal of the semigroup S, i. e. $A^m S \cap SA^n \subseteq A$. We prove that

$$A = \{A\}_{(m, 0)} \cap \{A\}_{(0, n)}.$$

By property *e)* of § 1 $\{A\}_{(m, 0)} = A \cup A^m S$, $\{A\}_{(0, n)} = A \cup SA^n$. By distributivity this implies

$$(A \cup A^m S) \cap (A \cup SA^n) = A \cup (A^m S \cap SA^n) = A,$$

as we stated.

Theorem 2.3. *Every (m, n)-quasiideal is an (m, n)-ideal.*

Proof. Let S be a semigroup, and let A be an (m, n)-quasiideal of S. Since $A^m SA^n \subseteq A^m S$ and $A^m SA^n \subseteq SA^n$ we obtain

$$A^m SA^n \subseteq A^m S \cap SA^n \subseteq A,$$

that is A is an (m, n)-ideal of S.

§ 3. The case of regular semigroups

Definition 3.1. A semigroup S is *regular* if to every element a of S there exists an element x in S so that $axa = a$.

Theorem 3.2. *In a regular semigroup every (m, n)-ideal is an (m, n)-quasiideal and conversely.*

Proof. Let S be a regular semigroup. We show that

$$A^m SA^n = A^m S \cap SA^n, \tag{5}$$

for every non-empty subset A of S. In proof of Theorem 2.3 we proved that $A^m SA^n \subseteq A^m S \cap SA^n$. Conversely, let x be an element of $A^m S \cap SA^n$. Then

$\left(\prod_{i=1}^{m} a_i\right) s = x = s' \cdot \prod_{j=1}^{n} a_j'$. Since S is regular, there exists an element y such that $xyx = x$. Thus $x = \left(\prod_{i=1}^{m} a_i\right) s \cdot y \cdot s' \cdot \prod_{j=1}^{n} a_j' \in A^m S A^n$, therefore

$$A^m S \cap S A^n \subseteq A^m S A^n,$$

i. e. holds (5), from which the theorem follows.

Corollary 3.3. *In a regular semigroup every biideal is a quasiideal, and conversely.*

Theorem 2.2 and Theorem 3.2 imply

Theorem 3.4. *A subset of a regular semigroup S is an (m, n)-ideal if and only if it is an intersection of an $(m, 0)$-ideal and a $(0, n)$-ideal of S.*

Now we prove the following

Lemma 3.5. *Let S be an arbitrary semigroup, and let M be an i^2-ideal of S. Denote $\bar{M}$ the two-sided ideal of S generated by M. Then $\bar{M}^3 \subseteq M$.*

Proof. Let M' be a two-sided ideal of S, containing M as two-sided ideal. Then

$$\bar{M}^3 \subseteq M' \bar{M} M' = M'(M \cup MS \cup SM \cup SMS)M' \subseteq M$$

because $\bar{M} = M \cup MS \cup SM \cup SMS$.

Theorem 3.6. *In a regular semigroup every i^k-ideal is a two-sided ideal (k is a positive integer).*

Proof. It is sufficient to prove that every i^2-ideal is a two-sided ideal. Let M be an i^2-ideal of a regular semigroup S, and let $\bar{M}$ be the two-sided ideal of S generated by M. From the Kovács—Iséki criteria of regularity (see [3] and [2]) it follows that $\bar{M}^2 = \bar{M}$. This and Lemma 3.5 imply $\bar{M} \subseteq M$, that is $\bar{M} = M$. Therefore M is a two-sided ideal of S, as we stated.

Remark 3.7. From the proof of Theorem 3.6 it can be seen that this theorem holds for every semigroup every two-sided ideal of which is idempotent.

References

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